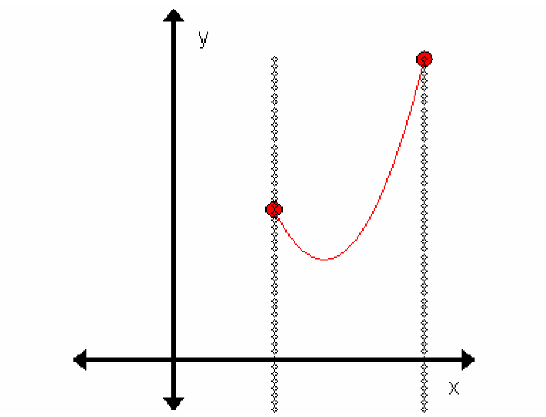
Unit 6:1A Extrema on an Interval

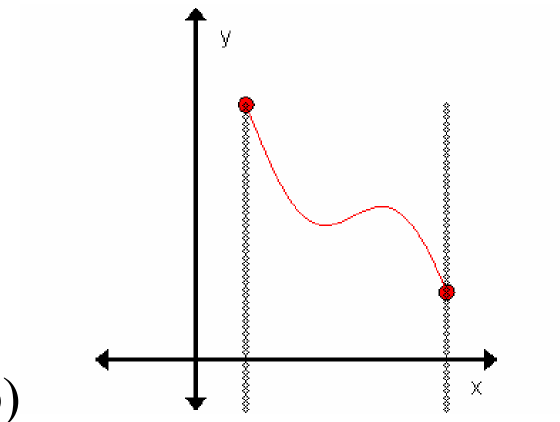
What is an extrema?

Describe the extrema on the following graphs on the indicated interval:

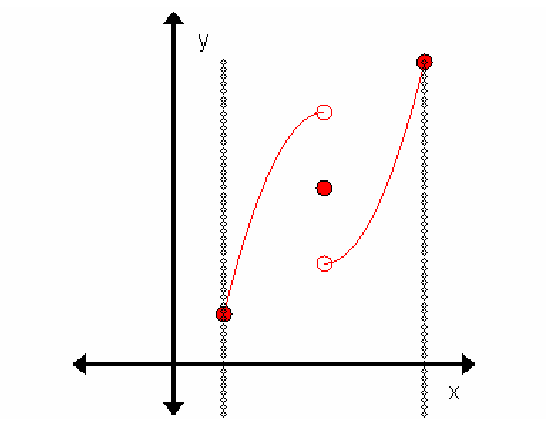
a)



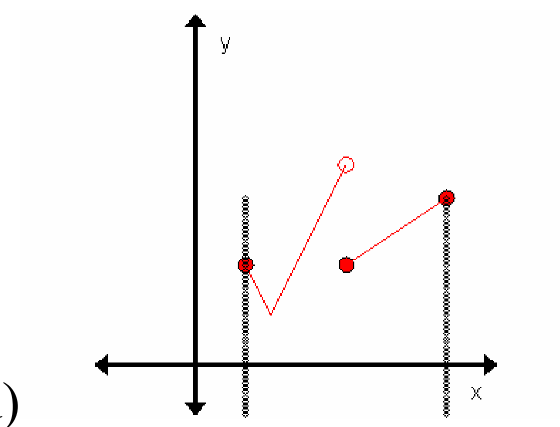
b)



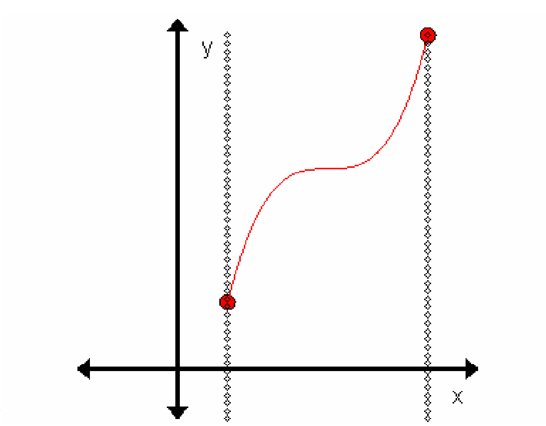
c)



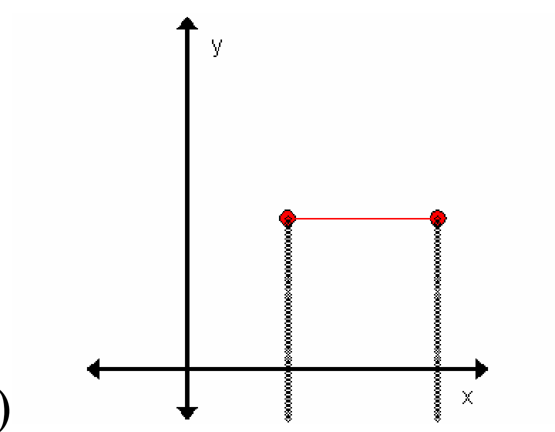
d)



e)



f)



6:1B

Definition of an extrema on an interval: A function f is said to have a maximum (minimum) on the interval I if there exist a number c in I, such that $f(c) \geq f(x)$ $\left(f(c) \leq f(x)\right)$ for all x in I.

Terminology: The maxima of a function in an interval [*a,b*] is the largest value that *f(x)* achieves. Thus the maximum is a "*y*" coordinate. We say that the maximum is achieved at a particular "*x*" coordinate (the *c* in the definition above).

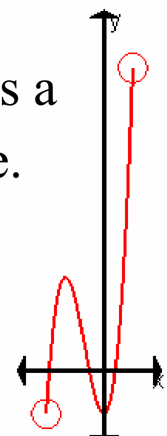
What is the maximum? → *y* coordinate

Were the maximum? → *x* coordinate

Examples:

- The function $y = x^2 + 1$ on the interval $[-1,1]$ has a minimum value of 1 at $x = 0$ and a maximum value of 2 at $x = -1$ and $x = 1$.

- The function $y = x^2(x+4) - 3$ on the interval $(-4, 2)$ has a
- minimum value of -3 at $x = 0$ and no maximum value.
-



When does a function have an extrema on an interval [a,b]?

> **Max/ Min Theorem:** If the function f is continuous on the closed interval [*a,b*], then both the maximum and minimum are achieved on the interval [*a,b*].

Relative Extrema: A function f is said to have a relative maximum (relative minimum) at *c*, if there exist an interval I containing *c*, such that $f(c) \geq f(x)$ $\left(f(c) \leq f(x)\right)$ for all x in I.

6:1C

Global Extrema: A value $f(c)$ is said to be a global maximum (global minimum) if $f(c) \geq f(x)$ $\left(f(c) \leq f(x)\right)$ for all x in the domain of f.

Examples:

- The function $y = \sqrt{25 - x^2}$ has a global minimum value of 0 at $x = -5$ and $x = 5$ and a global maximum value of 5 at $x = 0$.
- The function $y = \dfrac{1}{x^2 + 1}$ has no global minimum value and a global maximum value of 1 at $x = 0$.

Critical Numbers: A critical number of a function f is a value (x) in the domain of f, such that the derivative of f at x either does not exist or is equal to zero. A **critical point** is the coordinate $(x, f(x))$ where x is a critical number.

> **Extreme Theorem: (Critical Number Theorem)**
> All relative extrema occur at critical numbers.

Note this only works in one direction (extrema $\rightarrow$ critical number), but the converse is not always true (critical number $\not\rightarrow$ extrema).

Saddle Point (stationary point): A saddle point (or stationary point) of a function f is a coordinate $\left(x, f(x)\right)$ on the graph of f, such $f'(x) = 0$ but the derivative is of the same sign on both sides of $\left(x, f(x)\right)$. [Example: $y = x^3$ has a saddle point at $(0,0)$ .3]

6:1D

Finding extrema on closed interval:

1. Calculate $\frac{dy}{dx}$
2. Solve for when $\frac{dy}{dx} = 0$ or $\frac{dy}{dx}$ is undefined
3. Evaluate y at the values found above (at critical numbers) and at the endpoints.
4. The largest value above is the maximum, the smallest value above is the minimum.

Example: Find and identify the types of extrema of $g(x) = x^3 - 4x^2 + 4x - 2$ on the interval $[-1, 2.5]$

$$g'(x) = 3x^2 - 8x + 4$$
$$= (3x-2)(x-2)$$

critical numbers: $x = \frac{2}{3}, 2$

$$g(-1) = -9$$
$$g\left(\frac{2}{3}\right) = \frac{32}{27} \cong 1.185$$
$$g(2) = 0$$
$$g(2.5) = \frac{5}{8} = 0.625$$

Thus $g(x)$ has a maximum value of $\frac{32}{27}$ at $x = \frac{2}{3}$, and a minimum value of -9 at $x = -1$.

Example: Find and identify the types of extrema of $h(x) = x^{2/3}$ on the interval $[-1, 8]$.

$h'(x) = \frac{2}{3}x^{-1/3}$ is undefined at $x = 0$. Since $h(-1) = 1$, $h(0) = 0$, and $h(8) = 4$; on the interval $[-1, 8]$, h has a minimum of 0 at $x = 0$, and a maximum of 4 at $x = 8$.